N-Fractional Calculus of Some Multiple Power Functions (Conditions for Univalency of Functions and Applications)

Miyakoda, Tsuyako

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Tsuyako Miyakoda

Abstract

By using the thechnique of the fractional calculus, we have two types of representations for γ-th differintegral of the function $\frac{1}{(((z-b)^2-c)^2-d)}$. The N-fractional calculus to the same function are derived from the different way. One of them is derived with the use of

$$\begin{aligned}(f(z))_\gamma &= ((((z-b)^2-c)^2-d)^{-2}(((z-b)^2-c)^2-d))_\gamma \\ &= \sum_{s=0}^{\infty}\frac{\Gamma(\gamma+1)}{s!\Gamma(\gamma+1-s)}\left((((z-b)^2-c)^2-d)^{-2}\right)_{\gamma-s}(((z-b)^2-c)^2-d)_s.\end{aligned}$$

1 Introduction

We have shown one expression of the fractional calculus for the function $\frac{1}{(((z-b)^2-c)^2-d)}$. Now we aim at having an another expression of the fractional calculus for the same function which shows the same value as the another one.

We adopt the following definition of the fractional calculus.

(**I**) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D=\{D_-,D_+\}$, $C=\{C_-,C_+\}$, C_- be a curve along the cut joining two points z and $-\infty+iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty+iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$\begin{aligned}f_\nu &= (f)_\nu={}_C(f)_\nu \\ &= \frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z)^{\nu+1}} \quad (\nu\notin Z^-), \qquad (1)\end{aligned}$$

$$(f)_{-m}=\lim_{\nu\to-m}(f)_\nu \quad (m\in Z^+), \qquad (2)$$

where

$$-\pi \leq arg(\zeta - z) \leq \pi \ for \ C_-, \quad 0 \leq arg(\zeta - z) \leq 2\pi \ \ for \ C_+,$$

$$\zeta \neq z, \quad z \in C, \quad \nu \in R, \quad \Gamma; \ Gamma \ function,$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = (\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}}) \quad (\nu \notin Z^-), \quad (Refer\ to[1]) \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\alpha(N^\beta f) \ \ (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$) " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \ \ (\nu \in R). \tag{7}$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i)

$$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(log(z-c))_{\alpha} = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_{\alpha} := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

2 Preliminary

The following theorem is already reported by K. Nishimoto [12].

Theorem D. We have

(i)

$$(((z-b)^{\beta}-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_k\Gamma(\beta k-\alpha\beta+\gamma)}{k!\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{1}$$

$$(|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}| < \infty),$$

and

(ii)

$$(((z-b)^{\beta}-c)^{\alpha})_{n} = (-1)^n(z-b)^{\alpha\beta-n}\sum_{k=0}^{\infty}\frac{[-\alpha]_k[\beta k-\alpha\beta]_n}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{2}$$

$$(n \in Z_0^+, \quad |\frac{c}{(z-b)^{\beta}}| < 1),$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad with \quad [\lambda]_0 = 1,$$

(Pochhammer's Notation).

(II) The theorem below is already reported by K.Nishimoto(cf. J. Frac. Calc. Vol.31 (2007), pp.11-23)

Theorem E. We have

(i)

$$\left(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{\gamma}=e^{-i\pi\gamma}(z-b)^{\alpha\beta\delta-\gamma}$$
$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{m!k!\Gamma(\beta k-\alpha\beta(\delta-m))}\left(\frac{c}{(z-b)^{\beta}}\right)^k\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m \quad (3)$$

$$\left(\left|\frac{\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{\Gamma(\beta k-\alpha\beta(\delta-m))}\right|<\infty\right),$$

and

(ii)

$$\left(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{n}=(-1)^n(z-b)^{\alpha\beta\delta-n}$$
$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k[\beta k-\alpha\beta(\delta-m)]_n}{m!k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m \quad (4)$$

where

$$\left(n\in Z_0^+,\ \left|\frac{c}{(z-b)^{\beta}}\right|<1,\ \left|\frac{d}{(z-b)^{\alpha\beta}}\right|<1\right).$$

We apply this theorem to obtain some theorems for some algebraic functions.

3 N-Fractional Calculus of the Functions $f(z) = (((z-b)^2-c)^2-d)^{-1}$

In a previous paper (JFC vol.34,Nov.(2008),pp.11-22), the next Theorem is presented already.

Theorem 1. Let

$$f=f(z)=\frac{1}{((z-b)^2-c)^2-d}\qquad \left(((z-b)^2-c)^2-d\neq 0\right) \qquad (1)$$

$$G(k,m,\gamma)=\frac{[1]_m[2+2m]_k\Gamma(2k+4+4m+\gamma)}{m!k!\Gamma(2k+4+4m)}, \qquad (2)$$

$$S = S(z) = \frac{c}{(z-b)^2}, \qquad (|S| < 1) \tag{3}$$

and

$$T = T(z) = \frac{d}{(z-b)^4}, \qquad (|T| < 1) \tag{4}$$

we have then

(i)

$$(f)_\gamma = e^{-i\pi\gamma}(z-b)^{-4-\gamma} \sum_{m,k=0}^{\infty} G(k,m,\gamma) S^k T^m, \quad (\gamma \notin Z^-) \tag{5}$$

and

(ii)

$$(f)_n = (-1)^n (z-b)^{-4-n} \sum_{m,k=0}^{\infty} G(k,m,n) S^k T^m. \quad (n \in Z_0^+) \tag{6}$$

Here we have the new representation for $(f)_\gamma$ as follows.

Theorem 2. Let $f = f(z)$, $S = S(z)$ and $T = T(z)$ be the same as in Theorem 1,and we set W and H as follows,

$$W(\gamma, s) = \sum_{m,k=0}^{\infty} H(k,m,\gamma,s), \tag{7}$$

$$H(k,m,\gamma,s) = \frac{[2]_m [4+2m]_k \Gamma(2k+8+4m+\gamma-s)}{m!k!\Gamma(2k+8+4m)} S^k T^m. \tag{8}$$

Then we have

(i)

$$\begin{aligned}(f)_\gamma \;=\; & e^{-i\pi\gamma}(z-b)^{-4-\gamma}\{S((1-S^2)-T)W(\gamma,0) - 4\gamma(1-S)W(\gamma,1) \\ & +2\gamma(\gamma-1)(3-S)W(\gamma,2) - 4\gamma(\gamma-1)(\gamma-2)W(\gamma,3) \\ & +\gamma(\gamma-1)(\gamma-2)(\gamma-3)W(\gamma,4)\}, \quad (\gamma \notin Z^-) \end{aligned} \tag{9}$$

and

(ii)

$$
\begin{aligned}
(f)_n \;=\;& (-1)^n(z-b)^{-4-n}\{((1-S)^2-T)W(n,0)-4n(1-S)W(n,1)\\
&+2n(n-1)(3-S)W(n,2)-4n(n-1)(n-2)W(n,3)\\
&+n(n-1)(n-2)(n-3)W(n,4)\}. \quad (n\in Z_0^+) \qquad (10)
\end{aligned}
$$

Proof of (i). According to Lemma (iv), we have

$$
(f)_\gamma = ((((z-b)^2-c)^2-d)^{-2}\cdot(((z-b)^2-c)^2-d))_\gamma \qquad (11)
$$

$$
=\sum_{s=0}^{\infty}\frac{\Gamma(\gamma+1)}{s!\Gamma(\gamma+1-s)}((((z-b)-2^c)^2-d)^{-2})_{\gamma-s}(((z-b)^2-c)^2-d)_s \qquad (12)
$$

and applying Theorem E.(i) to

$$
((((z-b)^2-c)^2-d)^{-2})_{\gamma-s}, \qquad (13)
$$

we obtain

$$
\begin{aligned}
(f)_\gamma =& \sum_{s=0}^{4}\frac{\Gamma(\gamma+1)}{s!\Gamma(\gamma+1-s)}\{e^{-i\pi(\gamma-s)}(z-b)^{-8-\gamma+s}\\
&\times\sum_{m,k=0}^{\infty}\frac{[2]_m[4+2m]_k\Gamma(2k+8+4m+\gamma-s)}{m!k!\Gamma(2k+8+4m)}S^kT^m\}((((z-b)^2-c)^2-d)_s\\
=& e^{-i\pi\gamma}\{(z-b)^{-8-\gamma}(((z-b)^2-c)^2-d)W(\gamma,0)\\
&-4\gamma(z-b)^{-7-\gamma}((z-b)^3-c(z-b))W(\gamma,1)\\
&+2\gamma(\gamma-1)(z-b)^{-6-\gamma}(3(z-b)^2-c)W(\gamma,2)\\
&-4\gamma(\gamma-1)(\gamma-2)(z-b)^{-5-\gamma}(z-b)W(\gamma,3)\\
&+\gamma(\gamma-1)(\gamma-2)(\gamma-3)(z-b)^{-4-\gamma}W(\gamma,4)\}, \qquad (\gamma\notin Z^-). \qquad (14)
\end{aligned}
$$

We have the equation (9) from above equation directly.
Proof of (ii). We have the result by setting $\gamma = n$ in the equation (9).

Furthermore by setting $c = 0$ in Theorem 2, we can derive the following Corollary.

Corollary 1. We have

(i)

$$
\left(\frac{1}{(z-b)^4-d}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{-4-\gamma}\{(1-T)\sum_{m=0}^{\infty}H(0,m,\gamma,0)
$$

$$-4\gamma\sum_{m=0}^{\infty}H(0,m,\gamma,1)+6\gamma(\gamma-1)\sum_{m=0}^{\infty}H(0,m,\gamma,2)$$
$$-4\gamma(\gamma-1)(\gamma-2)\sum_{m=0}^{\infty}H(0,m,\gamma,3)$$
$$+\gamma(\gamma-1)(\gamma-2)(\gamma-3)\sum_{m=0}^{\infty}H(0,m,\gamma,4)\},\quad (\gamma\notin Z^-)\ (15)$$

and

(ii)

$$\left(\frac{1}{(z-b)^4-d}\right)_n=(-1)^n(z-b)^{-4-n}\{(1-T)\sum_{m=0}^{\infty}H(0,m,n,0)$$
$$-4n\sum_{m=0}^{\infty}H(0,m,n,1)+6n(n-1)\sum_{m=0}^{\infty}H(0,m,n,2)$$
$$-4n(n-1)(n-2)\sum_{m=0}^{\infty}H(0,m,n,3)$$
$$+n(n-1)(n-2)(n-3)\sum_{m=0}^{\infty}H(0,m,n,4)\}.\quad (n\in Z^-)\ (16)$$

4 Identities

We have the following identities with using W and H given in §3.

Theorem 3. We have

(i)

$$\sum_{m,k=0}^{\infty}\frac{[1]_m[2+2m]_k\Gamma(2k+4+4m+\gamma)}{m!k!\Gamma(2k+4+4m)}S^kT^m$$
$$=((1-S^2)-T)W(\gamma,0)-4\gamma(1-S)W(\gamma,1)$$
$$+2\gamma(\gamma-1)(3-S)W(\gamma,2)-4\gamma(\gamma-1)(\gamma-2)W(\gamma,3)$$
$$+\gamma(\gamma-1)(\gamma-2)(\gamma-3)W(\gamma,4),\qquad (\gamma\notin Z^-)\qquad (1)$$

and

(ii)

$$\sum_{m,k=0}^{\infty} \frac{[1]_m[2+2m]_k[2k+4+4m]_n}{m!k!} S^k T^m$$
$$= ((1-S)^2 - T)W(n,0) - 4n(1-S)W(n,1)$$
$$+2n(n-1)(3-S)W(n,2) - 4n(n-1)(n-2)W(n,3)$$
$$+n(n-1)(n-2)(n-3)W(n,4). \qquad (n \in Z_0^+). \qquad (2)$$

Proof. From Theorems 1 and 2 we can obtain above equations directly.

And by setting $c = 0$ in Theorem 3 we have the following collorary immediately.

Corollary 2.

(i)

$$\sum_{m=0}^{\infty} \frac{[1]_m \Gamma(4+4m+\gamma)}{m!\Gamma(4+4m)} T^m = (1-T)\sum_{m=0}^{\infty} H(0,m,\gamma,0) - 4\gamma \sum_{m=0}^{\infty} H(0,m,\gamma,1)$$
$$+6\gamma(\gamma-1)\sum_{m=0}^{\infty} H(0,m,\gamma,2) - 4\gamma(\gamma-1)(\gamma-2)\sum_{m=0}^{\infty} H(0,m,\gamma,3)$$
$$+\gamma(\gamma-1)(\gamma-2)(\gamma-3)\sum_{m=0}^{\infty} H(0,m,\gamma,4)\}, \qquad (\gamma \notin Z^-) \qquad (3)$$

and

(ii)

$$\sum_{m=0}^{\infty} \frac{[1]_m[4+4m]_n}{m!} T^m = (1-T)\sum_{m=0}^{\infty} H(0,m,n,0) - 4n\sum_{m=0}^{\infty} H(0,m,n,1)$$
$$+6n(n-1)\sum_{m=0}^{\infty} H(0,m,n,2) - 4n(n-1)(n-2)\sum_{m=0}^{\infty} H(0,m,n,3)$$
$$+n(n-1)(n-2)(n-3)\sum_{m=0}^{\infty} H(0,m,n,4)\}, \qquad (n \in Z^-) \qquad (4)$$

where $H(k,m,\gamma,s)$ is the same one given in Theorem 2.

5 A Special Case

In order to make sure of the formulation of Theorem 2, we consider the cace of $n = 1$. When $n = 1$, from Theorem 2 (ii), we have

$$\left(\frac{1}{((z-b)^2-c)^2-d}\right)_1 = -(z-b)^{-5}\{((1-S)^2-T)W(1,0)-4(1-S)W(1,1)\}. \tag{1}$$

And we noteice following relations,

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k}z^k = (1-z)^{-\lambda} \tag{2}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{[\lambda]_k k}{k!}T^k &= \sum_{k=0}^{\infty}\frac{[\lambda]_k}{(k-1)!}T^k \\ &= \sum_{k=0}^{\infty}\frac{[\lambda]_{k+1}}{k!}T^{k+1} \\ &= \lambda T\sum_{k=0}^{\infty}\frac{[\lambda+1]_k}{k}T^k = \lambda T(1-T)^{-1-\lambda}\end{aligned} \tag{3}$$

$$[\lambda]_{k+1} = \frac{\Gamma(\lambda+1+k)}{\Gamma(\lambda)} = \lambda[\lambda+1]_k \tag{4}$$

Then, we have the following relations with applying to the above euations.

$$W(1,0) = \sum_{m,k=0}^{\infty} H(k,m,1,0) \tag{5}$$

$$= \sum_{m,k=0}^{\infty}\frac{[2]_m[4+2m]_k[2k+8+4m]_1}{k!}S^kT^m \tag{6}$$

$$= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{[4+2m]_k(2k+8+4m)}{k!}S^k\right)\frac{[2]_m}{m!}T^m \tag{7}$$

$$= \sum_{m=0}^{\infty}\left(2\sum_{k=0}^{\infty}\frac{[4+2m]_k k}{k}S^k + 8\sum_{k=0}^{\infty}\frac{[4+2m]_k}{k!}S^k + 4m\sum_{k=0}^{\infty}\frac{[4+2m]_k}{k!}S^k\right)\frac{[2]_m}{m!}T^m$$

$$= \sum_{m=0}^{\infty}\left(2(4+2m)S(1-S)^{-5-2m} + 8(1-S)^{-4-2m} + 4m(1-S)^{-4-2m}\right)\frac{[2]_m}{m!}T^m$$

$$= S(1-S)^{-5}\left(8\sum_{m=0}^{\infty}\frac{[2]_m}{m!}(\frac{T}{(1-S)^2})^m + 4\sum_{m=0}^{\infty}\frac{[2]_m m}{m!}(\frac{T}{(1-S)^2})^m\right)$$

$$+8(1-S)^{-4}\sum_{m=0}^{\infty}\frac{[2]_m}{m!}(\frac{T}{(1-S)^2})^m+4(1-S)^{-4}\sum_{m=0}^{\infty}\frac{[2]_m m}{m!}(\frac{T}{(1-S)^2})^m \tag{8}$$

$$=S(1-S)^{-5}\Big(8(1-\frac{T}{(1-S)^2})^{-2}+8(\frac{T}{(1-S)^2})(1-\frac{T}{(1-S)^2})^{-3}\Big)$$

$$+8(1-S)^{-4}(1-\frac{T}{(1-S)^2})^{-2}+8(1-S)^{-4}(\frac{T}{(1-S)^2})(1-\frac{T}{(1-S)^2})^{-3} \tag{9}$$

$$=\frac{8(1-S)}{((1-S)^2-T)^3} \tag{10}$$

and

$$W(1,1)=\sum_{m,k=0}^{\infty}H(k,m,1,1)=\sum_{m,k=0}^{\infty}\frac{[2]_m[4+2m]_k}{m!k!}S^kT^m \tag{11}$$

$$=\sum_{m=0}^{\infty}\Big(\sum_{k=0}^{\infty}\frac{[4+2m]_k}{k!}S^k\Big)\frac{[2]_m}{m!}T^m=(1-S)^{-4}\sum_{m=0}^{\infty}\frac{[2]_m}{m!}\Big(\frac{T}{(1-S)^2}\Big)^m \tag{12}$$

$$=(1-S)^{-4}\Big(1-\frac{T}{(1-S)^2}\Big)^{-2}=\frac{1}{((1-S)^2-T)^2}. \tag{13}$$

Therefore

$$\Big(\frac{1}{((z-b)^2-c)^2-d}\Big)_1=-(z-b)^{-5}\{((1-S)^2-T)\frac{8(1-S)}{((1-S)^2-T)^3}$$

$$-4(1-S)\frac{1}{((1-S)^2-T)^2}\}$$

$$=-4(z-b)((z-b)^2-c)(((z-b)^2-c)^2-d)^{-2}. \tag{14}$$

Now this result is consistent with the one of the classical calculus of

$$\frac{d}{dz}(((z-b)^2-c)^2-d)^{-1}. \tag{15}$$

Here we confirm again the result for Theorem 1.

When $n=1$, from Theorem 1.(6), we have

$$\Big(\frac{1}{((z-b)^2-c)^2-d}\Big)_1=-(z-b)^{-5}\sum_{m,k=0}^{\infty}G(k,m,1)S^kT^m \tag{16}$$

$$=-(z-b)^{-5}\sum_{m,k=0}^{\infty}\frac{[1]_m[2+2m]_k[2k+4+4m]_1}{m!k!}S^kT^m \tag{17}$$

$$= -(z-b)^{-5} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m \times \sum_{k=0}^{\infty} \frac{[2+2m]_k (2k+4+4m)}{k!} S^k \tag{18}$$

$$= -(z-b)^{-5} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m \times \{2 \sum_{k=1}^{\infty} \frac{[2+2m]_k}{(k-1)!} S^k$$

$$= -(z-b)^{-5} \{2 \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m \{\sum_{k=1}^{\infty} \frac{[2+2m]_k}{(k-1)!} S^k\}$$

$$+4 \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m (1-S)^{-2-2m} + 4 \sum_{m=0}^{\infty} \frac{[1]_m m}{m!} T^m (1-S)^{-2-2m}\} \tag{19}$$

$$= -(z-b)^{-5} \{4 \frac{cX^2(X-c)}{((X-c)^2-d)^2} + 4 \frac{X^2}{(X-c)^2-d} + 4 \frac{X^2 d}{((X-c)^2-d)^2}\}$$

$$(where\ X = (z-b)^2) \tag{20}$$

$$= -(z-b)^{-5} \{\frac{X^3(X-c)}{((X-c)^2-d)^2}\} \tag{21}$$

Then we have

$$\left(\frac{1}{((z-b)^2-c)^2-d}\right)_1 = -4(z-b)^6((z-b)^2-c)(((z-b)^2-c)^2-d)^{-2}. \tag{22}$$

This result also coincides with the one cbtained by the classical calculus.

So we conclude that according to the definition of fractional differintegration , we have two representations for γ-th differintegrate of the function $\frac{1}{((z-b)^2-c)^2-d}$ by Theorem 1 and 2.

We can make sure that they have the same results as the classical result when the differential order is in the case of $n = 1$.

References

[1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5, (1996), Descartes Press, Koriyama, Japan.

[2] K. Nishimoto ; An Essence of Nishimnoto's Fractional Calculus (Calculus of the 21st Century); Integrals and Differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.

[3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^ν (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

[4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol. 6, Nov. (1994), 1 - 14.

[5] K. Nishimoto ; Ring and Field Produced from The Set of N- Fractional Calculus Operator, J. Frac Calc. Vol. 24, Nov. (2003),29 - 36.

[6] K. Nishimoto ; On the fractional calculus of functions $(a-z)^{\beta}$ and $\log(a-z)$, J. Frac. Calc. Vol. 3, May (1993), 19 - 27.

[7] K. Nishimoto and S. -T. Tu ; Fractional calculus of Psi functions (Generalized polygamma functions), J. Frac. Calc. Vol. 5, May (1994), 27 - 34.

[8] S. -T. Tu and K. Nishimoto ; On the fractional calculus of functions $(cz-a)^{\beta}$ and $\log(cz-a)$, J. Frac. Calc. Vol. 5, May (1994), 35 - 43.

[9] K. Nishimoto ; N-Fractional Calculus of the Power and Logarithmic Functions, and some Identities, J. Frac. Calc. Vol. 21, May (2002), 1 - 6.

[10] K. Nishimoto ; Some Theorems for N-Fractional Calculus of Logarithmic Functions, J. Frac. Calc. Vol. 21, May (2002), 7 - 12.

[11] K. Nishimoto ; N-Fractional Calculus of Products of Some Power Functions, J. Frac. Calc. Vol. 27, May (2005), 83 - 88.

[12] K. Nishimoto ; N-Fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol. 29, May (2006), 35 - 44.

[13] K. Nishimoto ; N-Fractional Calculus of Some Composite Algebraic Functions, J. Frac. Calc. Vol. 31, May (2007), 11 - 23.

[14] K. Nishimoto ; N-Fractional Calculus of Some Elementary Functions and Their Semi Differintegrations, J. Frac. Calc. Vol. 31, May (2007), 1 - 10.

[15] K. Nishimoto and T. Miyakoda ; N-Fractional Calculus and n-th Derivatives of Some Algebraic Functions, J. Frac. Calc. Vol. 31, May (2007), 53 - 62.

[16] T. Miyakoda ; N-Fractional Calculus of Certain Algebraic Functions, J. Frac. Calc. Vol. 31, May (2007), 63 - 76.

[17] K. Nishimoto ; N-Fractional Calculus and n-th Derivatives of Some Functions, J. Frac. Calc. Vol. 31, May (2007), 77 - 90.

[18] K. Nishimoto and T. Miyakoda ; N-Fractional Calculus of Some Multiple Power Functions and Some Identities, J. Frac. Calc. Vol. 34, Nov. (2008), 11 - 22.

[19] S.Moriguchi, K.Udagawa and S.HItotsumatsu ; Mathematical Formulae, Vol.1, Iwanami Zensho, (1957), Iwanami, Japan.

Tsuyako Miyakoda
Osaka Institute of Technology,
5-16-1, Ohmiya, Asahi-ku
Osaka 563 - 8585, Osaka,
JAPAN
Fax : + 81 - 6 - 6318 - 7485